

Matrix Representation of Bi-Periodic Jacobsthal Sequence

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Abstract

In this paper, we bring into light the matrix representation of bi-periodic Jacobsthal sequence, which we shall call the bi-periodic Jacobsthal Matrix sequence. We define it as

$$J_n = \begin{cases} bJ_{n-1} + 2J_{n-2}, & \text{if } n \text{ is even} \\ aJ_{n-1} + 2J_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2,$$

with initial conditions $J_0 = I$ identity matrix, $J_1 = \begin{pmatrix} b & 2\frac{b}{a} \\ 1 & 0 \end{pmatrix}$.

We obtained the n th general term of this new matrix sequence. By studying the properties of this new matrix sequence, the well-known Cassini or Simpson's formula was obtained. We then proceeded to find its generating function as well as the Binet formula. Some new properties and two summation formulas for this new generalized matrix sequence are also given.

Keywords: Bi-periodic Jacobsthal sequence; Generating function; Binet formula.

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1 Introduction

The increasing applications of integer sequences such as Fibonacci, Lucas, Jacobsthal, Jacobsthal Lucas, Pell etc in the various fields of science and arts can not be overemphasized. For example, the ratio of two consecutive Fibonacci numbers converges to what is widely known as the Golden ratio whose applications appear in many research areas, particularly in Physics, Engineering, Architecture, Nature and Art. Horadam[1]

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The same can easily be said for Jacobsthal sequence. For instance, it is known that Microcontrollers and other computers change the flow of execution of a program using conditional instructions. Along with branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction which boil down to being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 on 5 bits, 21 on 6 bits, 43 on 7 bits, 85 on 8 bits and continue in that order, which are exactly the Jacobsthal numbers [14].

Now, The classical Jacobsthal sequence $\{j_n\}_{n=0}^{\infty}$ which was named after the German mathematician Ernst Jacobsthal is defined recursively by the relation $j_n = j_{n-1} + 2j_{n-2}$ with initial conditions $j_0 = 0, j_1 = 1$. The other related sequence is the Jacobsthal Lucas sequence $\{c_n\}_{n=0}^{\infty}$ which satisfies the same recurrence relation, that is $c_n = c_{n-1} + 2c_{n-2}$ but with different initial conditions $c_0 = 2, c_1 = 1$. Applications of these two sequences to curves can be found in [13]

There are many generalization in literature on the above well-known integer sequences many of which can be found in our references. For example, in [6,7], Edson and Yayenie defined the bi-periodic Fibonacci sequence as

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2$$

with initial conditions $q_0 = 0, q_1 = 1$. After this, Bilgici [8] defined the bi-periodic Lucas sequence as

$$l_n = \begin{cases} al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd} \\ bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even} \end{cases} \quad n \geq 2$$

with initial conditions $l_0 = 2, l_1 = a$. Bilgici also established some relationships between the bi-periodic Fibonacci and Lucas numbers.

In [12], we defined the bi-periodic Jacobsthal sequence as

$$\hat{j}_n = \begin{cases} a\hat{j}_{n-1} + 2\hat{j}_{n-2}, & \text{if } n \text{ is even} \\ b\hat{j}_{n-1} + 2\hat{j}_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2,$$

with initial conditions $\hat{j}_0 = 0, \hat{j}_1 = 1$. In [15], we also brought into light bi-periodic Jacobsthal Lucas sequence $\{C_n\}_{n=0}^{\infty}$ as

$$C_n = \begin{cases} bC_{n-1} + 2C_{n-2}, & \text{if } n \text{ is even} \\ aC_{n-1} + 2C_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2.$$

with initial conditions $C_0 = 2, C_1 = a$. The direct relationship between the bi-periodic Jacobsthal and the bi-periodic Jacobsthal Lucas sequences were obtained as $C_n = 2\hat{j}_{n-1} + \hat{j}_{n+1}$ and $(ab + 8)\hat{j}_n = 2C_{n-1} + C_{n+1}$.

In [16], Coskun and Taskara defined the bi-periodic Fibonacci matrix sequence as

$$F_n(a, b) = \begin{cases} aF_{n-1}(a, b) + 2F_{n-2}(a, b), & \text{if } n \text{ is even} \\ bF_{n-1}(a, b) + 2F_{n-2}(a, b), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2,$$

with the initial conditions given as

$$F_0(a, b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_1(a, b) = \begin{pmatrix} b & \frac{b}{a} \\ 1 & 0 \end{pmatrix}.$$

They then obtained the n th general term of this matrix sequence as

$$F_n = \begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(n)} q_{n+1} & \frac{b}{a} q_n \\ q_n & \left(\frac{b}{a}\right)^{\varepsilon(n)} q_{n-1} \end{pmatrix},$$

where $\varepsilon(m)$ is the parity function which is defined as

$$\varepsilon(m) = \begin{cases} 0, & \text{if } m \text{ is even} \\ 1, & \text{if } m \text{ is odd} \end{cases}$$

In addition, the authors obtained the binet formula for this sequence as

$$J_n = A(\alpha^n - \beta^n) + B\left(\alpha^{2\lfloor \frac{n}{2} \rfloor + 2} - \beta^{2\lfloor \frac{n}{2} \rfloor + 2}\right)$$

where

$$A = \frac{(F_1(a, b) - bF_0(a, b))^{\varepsilon(n)} (aF_1(a, b) - F_0(a, b) - abF_0(a, b))^{1-\varepsilon(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor} (\alpha - \beta)} \quad \text{and} \quad B = \frac{b^{\varepsilon(n)} F_0(a, b)}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1} (\alpha - \beta)}$$

and $\alpha = \frac{ab + \sqrt{a^2b^2 + 8ab}}{2}$ and $\beta = \frac{ab - \sqrt{a^2b^2 + 8ab}}{2}$ are the roots of the characteristic equation $x^2 - abx - 2ab = 0$. Using the Binet formula, some summations for the bi-periodic Fibonacci matrix sequence were also given.

In the same way, In [11] Coskun, Yilmaz and Taskara defined the bi-periodic Lucas matrix sequence as

$$L_n(a, b) = \begin{cases} aL_{n-1}(a, b) + 2L_{n-2}(a, b), & \text{if } n \text{ is even} \\ bL_{n-1}(a, b) + 2L_{n-2}(a, b), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \quad (1)$$

with the initial conditions given as

$$L_0(a, b) = \begin{pmatrix} a & 2 \\ 2\frac{a}{b} & -a \end{pmatrix}, \quad L_1(a, b) = \begin{pmatrix} a^2 + 2\frac{a}{b} & a \\ \frac{a^2}{b} & 2\frac{a}{b} \end{pmatrix}.$$

They then obtained the n th general term of this matrix sequence as

$$L_n(a, b) = \begin{pmatrix} \left(\frac{a}{b}\right)^{\varepsilon(n)} l_{n+1} & l_n \\ \frac{a}{b} l_n & \left(\frac{a}{b}\right)^{\varepsilon(n)} l_{n-1} \end{pmatrix},$$

where $\varepsilon(m)$ is the parity function which is defined as before.

In addition, the authors obtained the binet formula for this sequence as

$$L_n = A\alpha^n + B\beta^n$$

where

$$A = \frac{bL_1(a, b) - \alpha L_0(a, b) - abL_0(a, b)}{b^{\varepsilon(n)}(ab)^{\lfloor \frac{n}{2} \rfloor}(\alpha - \beta)} \quad \text{and} \quad B = \frac{bL_1(a, b) - \beta L_0(a, b) - abL_0(a, b)}{b^{\varepsilon(n)}(ab)^{\lfloor \frac{n}{2} \rfloor}(\alpha - \beta)}$$

In this paper, as first in literature, we bring into light the matrix representation of bi-periodic Jacobsthal sequence, which we shall call the bi-periodic Jacobsthal Matrix sequence. We will then proceed to obtain the n th general term of this new matrix sequence. By studying the algebraic properties of this new matrix sequence, the well-known Cassini or Simpson's formula would be given. The generating function together with the Binet formula are also given. Some new properties as well as some summation formulas for this new generalized matrix sequence are also given.

Definition 1 For any two non-zero real numbers a and b , and any number n belonging to the set of natural numbers, the bi-periodic Jacobsthal matrix sequence denoted by $J_n(a, b)$ is defined recursively by

$$J_n(a, b) = \begin{cases} aJ_{n-1}(a, b) + 2J_{n-2}(a, b), & \text{if } n \text{ is even} \\ bJ_{n-1}(a, b) + 2J_{n-2}(a, b), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2,$$

with the initial conditions given as

$$J_0(a, b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_1(a, b) = \begin{pmatrix} b & 2\frac{b}{a} \\ 1 & 0 \end{pmatrix}.$$

For the brevity, we shall use J_n in place of $J_n(a, b)$.

Theorem 2 For any integer $n \geq 0$, we obtain the n th Jacobsthal matrix sequence as

$$J_n = \begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(n)} j_{n+1} & 2\frac{b}{a} j_n \\ j_n & 2\left(\frac{b}{a}\right)^{\varepsilon(n)} j_{n-1} \end{pmatrix}.$$

Proof. The proof is done by means of mathematical induction. We will start by noting from the classical Jacobsthal sequence j_n as defined in the introduction that $j_0 = 0$, $j_1 = 1$, $j_{-1} = \frac{1}{2}$ and $j_2 = a$. Hence the induction for $n = 0$ and $n = 1$ are respectively as follows

$$J_0 = \begin{pmatrix} j_1 & 2\frac{b}{a}j_0 \\ j_0 & 2j_{-1} \end{pmatrix} = I$$

■

$$J_1 = \begin{pmatrix} \left(\frac{b}{a}\right) j_2 & 2\frac{b}{a} j_1 \\ j_1 & 2j_0 \end{pmatrix} = \begin{pmatrix} b & 2\frac{b}{a} \\ 1 & 0 \end{pmatrix}$$

We now assume that the equation is true for $n = k$, where k is a positive integer, that is;

$$J_k = \begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(k)} j_{k+1} & 2\frac{b}{a} j_k \\ j_k & 2\left(\frac{b}{a}\right)^{\varepsilon(k)} j_{k-1} \end{pmatrix}$$

We will end the proof by showing that the equation also holds for $n = k + 1$; that is

$$\begin{aligned} J_{k+1} &= \begin{cases} aJ_k + 2J_{k-1}, & \text{if } k+1 \text{ is even} \\ bJ_k + 2J_{k-1}, & \text{if } k+1 \text{ is odd} \end{cases} \\ &= a^{\varepsilon(k)} b^{1-\varepsilon(k)} [J_k + 2J_{k-1}] \\ &= a^{\varepsilon(k)} b^{1-\varepsilon(k)} \begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(k)} j_{k+1} & 2\frac{b}{a} j_k \\ j_k & 2\left(\frac{b}{a}\right)^{\varepsilon(k)} j_{k-1} \end{pmatrix} + 2 \begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(k)} j_k & 2\frac{b}{a} j_{k-1} \\ j_{k-1} & 2\left(\frac{b}{a}\right)^{\varepsilon(k)} j_{k-2} \end{pmatrix} \\ &= \begin{cases} \begin{pmatrix} bj_{k+1} + 2\frac{b}{a} j_k & 2\left(\frac{b}{a}\right)^2 j_k + 4\frac{b}{a} j_{k-1} \\ bj_k + 2j_{k-1} & 2bj_{k=1} + 4\frac{b}{a} j_{k=2} \end{pmatrix} & k \text{ even} \\ \begin{pmatrix} a\frac{b}{a} j_{k+1} + 2j_k & 2a\frac{b}{a} j_k + 4\frac{b}{a} j_{k-1} \\ aj_k + 2j_{k-1} & 2a\frac{b}{a} j_{k=1} + 4j_{k=2} \end{pmatrix} & k \text{ odd} \end{cases} \\ &= \begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(k+1)} j_{k+2} & 2\frac{b}{a} j_{k+1} \\ j_{k+1} & 2\left(\frac{b}{a}\right)^{\varepsilon(k)} j_k \end{pmatrix}. \end{aligned}$$

Lemma 3 For any integer $m \geq 0$, we obtain

$$\begin{aligned} J_{2m} &= (ab + 4)J_{2m-2} - 4J_{2m-4}, \\ J_{2m+1} &= (ab + 4)J_{2m-1} - 4J_{2m-3}. \end{aligned}$$

Proof. The proof can easily be obtained by using the above definition of the bi-periodic Jacobsthal matrix sequence. ■

Theorem 4 For any positive integer n , we have

$$\det [J_n] = 2^n \left(-\frac{b}{a}\right)^{\varepsilon(n)}$$

Proof.

$$\det [J_0] = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

$$\det [J_1] = \det \begin{pmatrix} b & 2\frac{b}{a} \\ 1 & 0 \end{pmatrix} = -2\frac{b}{a}$$

$$\det [J_2] = \det \begin{pmatrix} ab+2 & 2b \\ a & 2 \end{pmatrix} = 4$$

$$\det [J_3] = \det \begin{pmatrix} a^2b+4b & 2b^2+4\frac{b}{a} \\ ab+2 & 2b \end{pmatrix} = -8\frac{b}{a}$$

$$\det [J_4] = \det \begin{pmatrix} a^2b^2+6ab+4 & 2ab^2+8b \\ a^2b+4a & 2ab+4 \end{pmatrix} = 16$$

$$\det [J_5] = \det \begin{pmatrix} a^2b^3+8ab^2+12b & 2ab^3+12b^2+8\frac{b}{a} \\ a^2b^2+6ab+4 & 2ab^2+8b \end{pmatrix} = -32\frac{b}{a}$$

$$\det [J_6] = \det \begin{pmatrix} a^3b^3+10a^2b^2+24ab+8 & 2a^2b^3+16ab^2+24b \\ a^3b^2+8a^2b+12a & 2a^2b^2+12ab+8 \end{pmatrix} = 64$$

... = the order continues.

■

Therefore the above procedure can be iterated as

$$\det [J_n] = \begin{cases} 2^n & \text{if } n \text{ even} \\ 2^n \left(-\frac{b}{a}\right) & \text{if } n \text{ odd} \end{cases}$$

which can be condensed using the parity function as

$$\det [J_n] = 2^n \left(-\frac{b}{a}\right)^{\varepsilon(n)}$$

which completes the proof

Corollary 5 (*CASSINI'S IDENTITY /SIMPSON'S FORMULA*)

Theorem 6 *This identity is obtain by a mere comparison of the determinants of the bi-periodic Jacobsthal matrix sequence with the determinant of its individual terms condensed together as shown in the immediate theorem above. Hence our casini's identity is given by*

$$2 \left(\frac{b}{a}\right)^{2\varepsilon(m)} j_{n-1}j_{n+1} - 2\frac{b}{a}j_n^2 = 2^n \left(-\frac{b}{a}\right)^{\varepsilon(n)}$$

which can be simplified as

$$\left(\frac{b}{a}\right)^{\varepsilon(m)} j_{n-1}j_{n+1} - \left(\frac{b}{a}\right)^{1-\varepsilon(m)} j_n^2 = (-1)^{\varepsilon(n)} 2^{n-1}$$

Theorem 7 (GENERATING FUNCTION)

The generating function for the bi-periodic Jacobsthal matrix sequence is obtained as

$$\sum_{m=0}^{\infty} J_m x^m = \frac{J_0 + J_1 x + [aJ_1 - (ab+2)J_0]x^2 + [2bJ_0 - 2J_1]x^3}{1 - (ab+4)x^2 + 4x^4}.$$

which is expressed in component form as

$$\sum_{m=0}^{\infty} J_m x^m = \frac{1}{1 - (ab+4)x^2 + 4x^4} \begin{pmatrix} 1 + bx - 2x^2 & 2\frac{b}{a}x + 2bx^2 - 4\frac{b}{a}x^3 \\ x + ax^2 - 2x^3 & 1 - (ab+2)x^2 + 2bx^3 \end{pmatrix}.$$

Proof. We divide the series into two parts

$$J(x) = \sum_{m=0}^{\infty} J_m x^m = \sum_{m=0}^{\infty} J_{2m} x^{2m} + \sum_{m=0}^{\infty} J_{2m+1} x^{2m+1}.$$

We simplify the even part of the above series as follows

$$J_0(x) = \sum_{m=0}^{\infty} J_{2m} x^{2m} = J_0 + J_2 x^2 + \sum_{m=2}^{\infty} J_{2m} x^{2m}$$

By multiplying through by $(ab+4)x^2$ and $4x^4$ respectively, we have

$$(ab+4)x^2 J_0(x) = (ab+4)J_0 x^2 + (ab+4) \sum_{m=2}^{\infty} J_{2m-2} x^{2m}$$

and

$$4x^4 J_0(x) = 4 \sum_{m=2}^{\infty} J_{2m-4} x^{2m}.$$

Hence it follows that,

$$\begin{aligned} [1 - (ab+4)x^2 + 4x^4] J_0(x) &= J_0 + J_2 x^2 - (ab+4)J_0 x^2 \\ &\quad + \sum_{m=2}^{\infty} [J_{2m} - (ab+4)J_{2m-2} + 4J_{2m-4}] x^{2m} \end{aligned}$$

By using Lemma 1, we obtained that;

$$J_0(x) = \frac{J_0 + J_2 x^2 - (ab+4)J_0 x^2}{1 - (ab+4)x^2 + 4x^4}.$$

Similarly, the odd part of the above series is simplified as follows

$$J_1(x) = \sum_{m=0}^{\infty} J_{2m+1} x^{2m+1} = J_1 x + J_3 x^3 + \sum_{m=2}^{\infty} J_{2m+1} x^{2m+1}$$

By multiplying through by $(ab + 4)x^2$ and $4x^4$ respectively, we obtain

$$(ab + 4)x^2 J_1(x) = (ab + 4)J_1(a, b)x^3 + (ab + 4)\sum_{m=2}^{\infty} J_{2m-1}(a, b)x^{2m+1}.$$

and

$$4x^4 J_1(x) = 4\sum_{m=2}^{\infty} J_{2m-3}x^{2m+1}.$$

Hence it follows that,

$$\begin{aligned} [1 - (ab + 4)x^2 + 4x^4] J_1(x) &= J_1x + J_3x^3 - (ab + 4)J_1x^3 \\ &\quad + \sum_{m=2}^{\infty} [J_{2m+1} - (ab + 4)J_{2m-1} + 4J_{2m-3}]x^{2m+1} \end{aligned}$$

By using Lemma 1, we obtained that;

$$J_1(x) = \frac{J_1x + J_3x^3 - (ab + 4)J_1x^3}{1 - (ab + 4)x^2 + 4x^4}.$$

By combining the two results $[J(x) = J_0(x) + J_1(x)]$, we have

$$J(x) = \frac{J_0 + J_2x^2 - (ab + 4)J_0x^2 + J_1x + J_3x^3 - (ab + 4)J_1x^3}{1 - (ab + 4)x^2 + 4x^4}.$$

which can be simplified using definition [1] as

$$J(x) = \frac{J_0 + J_1x + [aJ_1 - (ab + 2)J_0]x^2 + [2bJ_0 - 2J_1]x^3}{1 - (ab + 4)x^2 + 4x^4}.$$

By substituting the matrix components of J_0 and J_1 as given in definition [1] and simplifying, we obtain

$$J(x) = \frac{1}{1 - (ab + 4)x^2 + 4x^4} \begin{pmatrix} 1 + bx - 2x^2 & 2\frac{b}{a}x + 2bx^2 - 4\frac{b}{a}x^3 \\ x + ax^2 - 2x^3 & 1 - (ab + 2)x^2 + 2bx^3 \end{pmatrix}.$$

which completes the proof. ■

We would like to show another proof of this theorem.

(2) The generating function for $J(x)$ is represented in power series by

$$J(x) = \sum_{m=0}^{\infty} J_mx^m = J_0 + J_1x + \dots + J_kx^k + \dots$$

By multiplying through this series by bx and $2x^2$ respectively, we obtain

$$bxJ(x) = b\sum_{m=0}^{\infty} J_mx^{m+1} = b\sum_{m=1}^{\infty} J_{m-1}x^m$$

and,

$$2x^2 J(x) = 2 \sum_{m=0}^{\infty} J_m x^{m+2} = 2 \sum_{m=2}^{\infty} J_{m-2} x^m.$$

therefore we have

$$\begin{aligned} (1 - bx - 2x^2)J(x) &= J_0 + xJ_1 + bxJ_0 \\ &\quad + \sum_{m=2}^{\infty} (J_m - bJ_{m-1} - 2J_{m-2})x^m \\ &= J_0 + xJ_1 + bxJ_0 \\ &\quad + \sum_{m=1}^{\infty} (J_{2m} - bJ_{2m-1} - 2J_{2m-2})x^{2m} \end{aligned}$$

From Lemma 1, $J_{2m} = aJ_{2m-1} + 2J_{2m-2}$, therefore

$$(1 - bx - 2x^2)J(x) = J_0 + xJ_1 + bxJ_0 + \sum_{m=1}^{\infty} (a - b)J_{2m-1}x^{2m}.$$

$$(1 - bx - 2x^2)J(x) = J_0 + xJ_1 + bxJ_0 + (a - b)x \sum_{m=1}^{\infty} J_{2m-1}x^{2m-1}.$$

Now lets define $j(x)$ as

$$j(x) = \sum_{m=1}^{\infty} J_{2m-1}x^{2m-1}.$$

Simplifying $j(x)$ in the same way as above and using lemma 1 gives;

$$\begin{aligned} (1 - (ab + 4)x^2 + 4x^4)j(x) &= \sum_{m=1}^{\infty} J_{2m-1}x^{2m-1} - (ab + 4) \sum_{m=2}^{\infty} J_{2m-3}x^{2m-1} \\ &\quad + 4 \sum_{m=3}^{\infty} J_{2m-5}x^{2m-1} \\ &= (J_1x + J_3x^3) - (ab + 4)J_1x^3 \\ &\quad + \sum_{m=3}^{\infty} (J_{2m-1} - (ab + 4)J_{2m-3} + 4J_{2m-5})x^{2m-1} \\ &= J_1x + J_3x^3 - (ab + 4)J_1x^3 + 0. \\ j(x) &= \frac{J_1x + J_3x^3 - (ab + 4)J_1x^3}{1 - (ab + 4)x^2 + 4x^4} \end{aligned}$$

Plugging $j(x)$ into $J(x)$ above gives

$$(1 - bx - 2x^2)J(x) = J_0 + xJ_1 + bxJ_0 + (a - b)x \left(\frac{J_1x + J_3x^3 - (ab + 4)J_1x^3}{1 - (ab + 4)x^2 + 4x^4} \right)$$

Simplifying this using the basic rules and properties of algebra, we obtain

$$J(x) = \frac{J_0 + J_2x^2 - (ab+4)J_0x^2 + J_1x + J_3x^3 - (ab+4)J_1x^3}{1 - (ab+4)x^2 + 4x^4}.$$

which simplifies as

$$J(x) = \frac{J_0 + J_1x + [aJ_1 - (ab+2)J_0]x^2 + [2bJ_0 - 2J_1]x^3}{1 - (ab+4)x^2 + 4x^4}.$$

Similarly the component form is obtained as.

$$J(x) = \frac{1}{1 - (ab+4)x^2 + 4x^4} \begin{pmatrix} 1 + bx - 2x^2 & 2\frac{b}{a}x + 2bx^2 - 4\frac{b}{a}x^3 \\ x + ax^2 - 2x^3 & 1 - (ab+2)x^2 + 2bx^3 \end{pmatrix}.$$

which completes the proof.

Theorem 8 (BINET FORMULA)

For every n belonging to the set of natural numbers, the Binet formula for the bi-periodic Jacobsthal matrix sequence is given by.

$$J_n = A(\alpha^n - \beta^n) + B\left(\alpha^{2\lfloor \frac{n}{2} \rfloor + 2} - \beta^{2\lfloor \frac{n}{2} \rfloor + 2}\right).$$

Where

$$\begin{aligned} A &= \frac{(J_1 - bJ_0)^{\epsilon(n)} (aJ_1 - 2J_0 - abJ_0)^{1-\epsilon(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor} (\alpha - \beta)} \quad \text{and} \\ B &= \frac{b^{\epsilon(n)} J_0}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1} (\alpha - \beta)} \end{aligned}$$

The matrices A and B are expressed in component form as

$$\begin{aligned} A &= \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor} (\alpha - \beta)} \left\{ \begin{pmatrix} 0 & 2\frac{b}{a} \\ 1 & -b \end{pmatrix}^{\epsilon(n)} \begin{pmatrix} -2 & 2b \\ a & -2 - ab \end{pmatrix}^{1-\epsilon(n)} \right\} \\ \text{and } B &= \frac{b^{\epsilon(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1} (\alpha - \beta)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Proof.

$$\Phi(x) = J_0 + J_1x + [aJ_1 - (ab+2)J_0]x^2 + [2bJ_0 - 2J_1]x^3$$

■

Using partial fraction decomposition, we split $J(x)$ as

$$\begin{aligned} J(x) &= \frac{\Phi(x)}{1 - (ab + 4)x^2 + 4x^4} \\ &= \frac{1}{4} \left[\frac{Ax + B}{\left(x^2 - \frac{\alpha+2}{4}\right)} + \frac{Cx + D}{\left(x^2 - \frac{\beta+2}{4}\right)} \right] \end{aligned}$$

By solving for the constants A, B, C and D above, we express $J(x)$ in partial fraction as

$$\frac{1}{4(\alpha - \beta)} \left[\frac{\left\{ \begin{array}{l} x \{2\alpha [bJ_0 - J_1] + 4bJ_0\} \\ + \alpha \{aJ_1 - 2J_0 - abJ_0\} \\ + 2aJ_1 - 2abJ_0 \end{array} \right\}}{\left(x^2 - \frac{\alpha+2}{4}\right)} + \frac{\left\{ \begin{array}{l} x \{2\beta [J_1 - bJ_0] - 4bJ_0\} \\ + \beta \{abJ_0(a, b) + 2J_0(a, b) - aJ_1\} \\ + 2abJ_0 - 2aJ_1 \end{array} \right\}}{\left(x^2 - \frac{\beta+2}{4}\right)} \right]$$

The Maclaurin series expansion of the function $\frac{Ax+B}{x^2-C}$ is expressed in the form

$$\frac{Ax + B}{x^2 - C} = - \sum_{n=0}^{\infty} AC^{-n-1} x^{2n+1} - \sum_{n=0}^{\infty} BC^{-n-1} x^{2n}$$

Hence $J(x)$ can be expanded and simplified as

$$J(x) = \frac{1}{4(\alpha - \beta)} \left[\begin{aligned} & - \sum_{n=0}^{\infty} \{2\alpha bJ_0 - J_1 + 4bJ_0\} \left(\frac{\alpha+2}{4}\right)^{-n-1} x^{2n+1} \\ & - \sum_{n=0}^{\infty} \left\{ \begin{array}{l} \alpha [aJ_1 - 2J_0 - abJ_0] \\ + 2aJ_1 - 2abJ_0 \end{array} \right\} \left(\frac{\alpha+2}{4}\right)^{-n-1} x^{2n} + \\ & - \sum_{n=0}^{\infty} \left\{ \begin{array}{l} 2\beta [J_1 - bJ_0] \\ - 4bJ_0 \end{array} \right\} \left(\frac{\beta+2}{4}\right)^{-n-1} x^{2n+1} \\ & - \sum_{n=0}^{\infty} \left\{ \begin{array}{l} \beta \{abJ_0 + 2J_0\} \\ - aJ_1 \\ + 2abJ_0 - 2aJ_1 \end{array} \right\} \left(\frac{\beta+2}{4}\right)^{-n-1} x^{2n} + \end{aligned} \right]$$

We obtain the even part of $J(x)$ as

$$\frac{-1}{4(\alpha - \beta)} \sum_{n=0}^{\infty} \left[\begin{aligned} & \left\{ \begin{array}{l} \alpha [aJ_1 - 2J_0 - abJ_0] \\ + 2aJ_1 - 2abJ_0 \end{array} \right\} \frac{4^{n+1}}{(\alpha+2)^{n+1}} \\ & + \left\{ \begin{array}{l} \beta [abJ_0 + 2J_0 - aJ_1] \\ + 2abJ_0 - 2aJ_1 \end{array} \right\} \frac{4^{n+1}}{(\beta+2)^{n+1}} \end{aligned} \right] x^{2n}$$

Which can be simplified as

$$\frac{-4^n}{(\alpha - \beta)} \sum_{n=0}^{\infty} \left[\frac{(\beta + 2)^{n+1} \left\{ \begin{array}{l} \alpha [aJ_1 - 2J_0 - abJ_0] \\ + 2aJ_1 - 2abJ_0 \end{array} \right\} + (\alpha + 2)^{n+1} \left\{ \begin{array}{l} \beta \{abJ_0 + 2J_0 - aJ_1\} \\ + 2abJ_0 - 2aJ_1 \end{array} \right\}}{(\alpha + 2)^{n+1} (\beta + 2)^{n+1}} \right] x^{2n}$$

From the identity that $(\alpha + 2)(\beta + 2) = 4$, we have

$$\frac{1}{4(\alpha - \beta)} \sum_{n=0}^{\infty} \left[\left\{ \begin{array}{c} 2\beta(\beta + 2)^n [aJ_1 - 2J_0 - abJ_0] \\ -(\beta + 2)^{n+1} [2aJ_1 - 2abJ_0] \end{array} \right\} + \left\{ \begin{array}{c} 2\alpha(\alpha + 2)^n \{abJ_0 + 2J_0 - aJ_1\} \\ -(\alpha + 2)^{n+1} [2abJ_0 - 2aJ_1] \end{array} \right\} \right] x^{2n}$$

by using the identity $(\alpha + 2) = \frac{\alpha^2}{ab}$, we get

$$\frac{2}{4(\alpha - \beta)} \sum_{n=0}^{\infty} \left(\frac{1}{ab} \right)^{n+1} \left\{ \begin{array}{c} [aJ_1 - 2J_0 - abJ_0] [ab(\beta^{2n+1} - \alpha^{2n+1})] \\ + [aJ_1 - abJ_0] [\alpha^{2n+2} - \beta^{2n+2}] \end{array} \right\} x^{2n}$$

Also, making use of the identity $ab = \alpha + \beta$ gives

$$\frac{1}{(\alpha - \beta)} \sum_{n=0}^{\infty} \left\{ \begin{array}{c} (aJ_1 - 2J_0 - abJ_0) \left\{ \frac{\alpha^{2n} - \beta^{2n}}{(ab)^n(\alpha - \beta)} \right\} \\ + J_0 \left\{ \frac{\alpha^{2n+2} - \beta^{2n+2}}{(ab)^{n+1}(\alpha - \beta)} \right\} \end{array} \right\} x^{2n}$$

In the same way, the odd part of $J(x)$ is obtained as

$$\frac{-4^{n+1}}{4(\alpha - \beta)} \sum_{n=0}^{\infty} \left[\{2\alpha[bJ_0 - J_1] + 4bJ_0\} \frac{4^{n+1}}{(\alpha + 2)^{n+1}} + \{2\beta[J_1 - bJ_0] - 4bJ_0\} \frac{4^{n+1}}{(\beta + 2)^{n+1}} \right] x^{2n+1}$$

which can be simplified as

$$\frac{-4^{n+1}}{4(\alpha - \beta)} \sum_{n=0}^{\infty} \left[\frac{(\beta + 2)^{n+1} \{2\alpha[bJ_0 - J_1] + 4bJ_0\} + (\alpha + 2)^{n+1} \{2\beta[J_1 - bJ_0] - 4bJ_0\}}{(\alpha + 2)^{n+1} (\beta + 2)^{n+1}} \right] x^{2n+1}$$

$\beta + 2 = -\frac{\beta}{\alpha}$, implies gives

$$\frac{-4}{4(\alpha - \beta)} \sum_{n=0}^{\infty} \left\{ \begin{array}{c} [bJ_0 - J_1] [\alpha(\alpha + 2)^n - \beta(\beta + 2)^n] + \\ bJ_0 [(\beta + 2)^{n+1} - (\alpha + 2)^{n+1}] \end{array} \right\} x^{2n+1}$$

with $(\alpha + 2) = \frac{\alpha^2}{ab}$, we simplify the above expression as

$$\frac{-1}{(\alpha - \beta)} \sum_{n=0}^{\infty} \left(\frac{1}{ab} \right)^{n+1} \left\{ \begin{array}{c} ab[bJ_0 - J_1] (\alpha^{2n+1} - \beta^{2n+1}) \\ - bJ_0 (\alpha^{2n+2} - \beta^{2n+2}) \end{array} \right\} x^{2n+1}$$

This can be further expanded and simplified as

$$\sum_{n=0}^{\infty} \left\{ (J_1 - bJ_0) \left[\frac{\alpha^{2n+1} - \beta^{2n+1}}{(ab)^n(\alpha - \beta)} \right] + bJ_0 \left[\frac{\alpha^{2n+2} - \beta^{2n+2}}{(ab)^{n+1}(\alpha - \beta)} \right] \right\} x^{2n+1}$$

Now the even and the odd expressions obtained can be condensed by means of the parity function [...] as

$$J(x) = \sum_{n=0}^{\infty} \left\{ \begin{array}{c} (J_1 - bJ_0)^{\epsilon(n)} (aJ_1 - 2J_0 - abJ_0)^{1-\epsilon(n)} \left\{ \frac{\alpha^n - \beta^n}{(ab)^{\lfloor \frac{n}{2} \rfloor} (\alpha - \beta)} \right\} \\ + b^{\epsilon(n)} J_0 \left\{ \frac{\alpha^{2(\lfloor \frac{n}{2} \rfloor + 1)} - \beta^{2(\lfloor \frac{n}{2} \rfloor + 1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1} (\alpha - \beta)} \right\} \end{array} \right\} x^n$$

Therefore compared with $J(x) = \sum_{n=1}^{\infty} J_n x^n$ we obtain our Binet formula as

$$J_n = A(\alpha^n - \beta^n) + B\left(\alpha^{2\lfloor \frac{n}{2} \rfloor + 2} - \beta^{2\lfloor \frac{n}{2} \rfloor + 2}\right)$$

where

$$A = \frac{(J_1 - bJ_0)^{\epsilon(n)} (aJ_1 - 2J_0 - abJ_0)^{1-\epsilon(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor} (\alpha - \beta)} \quad \text{and} \quad B = \frac{b^{\epsilon(n)} J_0}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1} (\alpha - \beta)}.$$

Theorem 9 (SUMMATION FORMULA)

For any positive integer n , we have

$$\sum_{k=0}^{n-1} J_k = \frac{J_n(1 - a^{\xi(n)} b^{1-\xi(n)}) + 2J_{n-1}(1 - a^{1-\xi(n)} b^{\xi(n)}) + J_1(a-1) + J_0(2b - ab - 1)}{1 - ab}.$$

Proof. If n is even, we have

$$\begin{aligned} \sum_{k=0}^{n-1} J_k &= \sum_{k=0}^{\frac{n-2}{2}} J_{2k} + \sum_{k=0}^{\frac{n-2}{2}} J_{2k+1} \\ &= \sum_{k=0}^{\frac{n-2}{2}} \frac{aJ_1 - 2J_0 - abJ_0}{(ab)^k} \frac{\alpha^{2k} - \beta^{2k}}{\alpha - \beta} + \sum_{k=0}^{\frac{n-2}{2}} \frac{J_0}{(ab)^{k+1}} \frac{\alpha^{2k+2} - \beta^{2k+2}}{\alpha - \beta} \\ &\quad + \sum_{k=0}^{\frac{n-2}{2}} \frac{J_1 - bJ_0}{(ab)^k} \frac{\alpha^{2k+1} - \beta^{2k+1}}{\alpha - \beta} + \sum_{k=0}^{\frac{n-2}{2}} \frac{bJ_0}{(ab)^{k+1}} \frac{\alpha^{2k+2} - \beta^{2k+2}}{\alpha - \beta} \end{aligned}$$

$$\begin{aligned} \sum_{k=0}^{n-1} J_k &= \frac{aJ_1 - 2J_0 - abJ_0}{(ab)^{\frac{n}{2}-1} (\alpha - \beta)} \left[\frac{\alpha^n - (ab)^{\frac{n}{2}}}{(\alpha^2 - ab)} - \frac{\beta^n - (ab)^{\frac{n}{2}}}{(\beta^2 - ab)} \right] \\ &\quad + \frac{J_0}{(ab)^{\frac{n}{2}} (\alpha - \beta)} \left[\frac{\alpha^{n+2} - \alpha^2 (ab)^{\frac{n}{2}}}{(\alpha^2 - ab)} - \frac{\beta^{n+2} - \beta^2 (ab)^{\frac{n}{2}}}{(\beta^2 - ab)} \right] \\ &\quad + \frac{J_1 - bJ_0}{(ab)^{\frac{n}{2}-1} (\alpha - \beta)} \left[\frac{\alpha^{n+1} - \alpha (ab)^{\frac{n}{2}}}{(\alpha^2 - ab)} - \frac{\beta^{n+1} - \beta (ab)^{\frac{n}{2}}}{(\beta^2 - ab)} \right] \\ &\quad + \frac{bJ_0}{(ab)^{\frac{n}{2}} (\alpha - \beta)} \left[\frac{\alpha^{n+2} - \alpha^2 (ab)^{\frac{n}{2}}}{(\alpha^2 - ab)} - \frac{\beta^{n+2} - \beta^2 (ab)^{\frac{n}{2}}}{(\beta^2 - ab)} \right]. \end{aligned}$$

$$\begin{aligned}
&= \frac{(aJ_1 - 2J_0 - abJ_0)}{(ab)^{\frac{n}{2}+1}(\alpha - \beta)(1 - ab)} \left[4a^2b^2(\alpha^{n-2} - \beta^{n-2}) - ab(\alpha^n - \beta^n) + (ab)^{\frac{n}{2}}(\alpha^2 - \beta^2) \right] \\
&+ \frac{J_0}{(ab)^{\frac{n}{2}+2}(\alpha - \beta)(1 - ab)} \left[4a^2b^2(\alpha^n - \beta^n) - ab(\alpha^{n+2} - \beta^{n+2}) + (ab)^{\frac{n}{2}+1}(\alpha^2 - \beta^2) \right] \\
&+ \frac{J_1 - bJ_0}{(ab)^{\frac{n}{2}+1}(\alpha - \beta)(1 - ab)} \left[4a^2b^2(\alpha^{n-1} - \beta^{n-1}) - ab(\alpha^{n+1} - \beta^{n+1}) - (ab)^{\frac{n}{2}+1}(\alpha - \beta) \right] \\
&+ \frac{bJ_0}{(ab)^{\frac{n}{2}+2}(\alpha - \beta)(1 - ab)} \left[4a^2b^2(\alpha^n - \beta^n) - ab(\alpha^{n+2} - \beta^{n+2}) + (ab)^{\frac{n}{2}+1}(\alpha^2 - \beta^2) \right] \\
&= \frac{-J_{n+1} - J_n + 4J_{n-1} + 4J_{n-2} + J_1(a-1) + J_0(2b - ab - 1)}{1 - ab} \\
&= \frac{-J_{n+1} - J_n + 4J_{n-1} + 4J_{n-2} + J_1(a-1) + J_0(2b - ab - 1)}{1 - ab} \\
&= \frac{J_n(1-b) + J_{n-1}(2-2a) + J_1(a-1) + J_0(2b - ab - 1)}{1 - ab}
\end{aligned}$$

similarly if n is odd, we obtain

$$\begin{aligned}
\sum_{k=0}^{n-1} J_k &= \sum_{k=0}^{\frac{n-1}{2}} J_{2k} + \sum_{k=0}^{\frac{n-3}{2}} J_{2k+1} \\
&= \frac{-J_{n+1} - J_n + 4J_{n-1} + 4J_{n-2} + J_1(a-1) + J_0(2b - ab - 1)}{1 - ab} \\
&= \frac{J_n(1-a) + J_{n-1}(2-2b) + J_1(a-1) + J_0(2b - ab - 1)}{1 - ab}
\end{aligned}$$

Hence putting the two results together by means of the parity function gives

$$\sum_{k=0}^{n-1} J_k = \frac{J_n(1 - a^{\xi(n)}b^{1-\xi(n)}) + 2J_{n-1}(1 - a^{1-\xi(n)}b^{\xi(n)}) + J_1(a-1) + J_0(2b - ab - 1)}{1 - ab}.$$

■

Theorem 10 (SUMMATION FORMULA)

For any positive integer n , we have

$$\begin{aligned}
\sum_{k=0}^{n-1} \frac{J_k}{x^k} &= \frac{J_n(2 - x - a^{\xi(n)}b^{1-\xi(n)}x) + 2J_{n-1}(2 - a^{1-\xi(n)}b^{\xi(n)}x - x)}{x^2 - (ab+4)x + 4} \\
&+ \frac{x^2(J_1 - bJ_0) + x(-2J_1 + 3bJ_0 + aJ_1 - J_0 - abJ_0)}{x^2 - (ab+4)x + 4}.
\end{aligned}$$

Proof. The proof is obtained in a similar fashion as the above theorem. ■

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